

Euler Rotations

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The rotation matrix $\mathbf{R}_{T \rightarrow W}$ corresponding to the SAE standard for Euler angle rotation sequence (first yaw (θ_z), then pitch (θ_y), then roll (θ_x)), is given by:

$$\mathbf{R}_{T \rightarrow W} = \begin{bmatrix} cz & -sz & 0 \\ sz & cz & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} cy & 0 & sy \\ 0 & 1 & 0 \\ -sy & 0 & cy \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx & -sx \\ 0 & sx & cx \end{bmatrix}$$

(rotation $\mathbf{R}_z$) (rotation $\mathbf{R}_y$) (rotation $\mathbf{R}_x$)

$$\mathbf{R}_{T \rightarrow W} = \begin{bmatrix} cy \cdot cz & sx \cdot sy \cdot cz - cx \cdot sz & cx \cdot sy \cdot cz + sx \cdot sz \\ cy \cdot sz & sx \cdot sy \cdot sz + cx \cdot cz & cx \cdot sy \cdot sz - sx \cdot cz \\ -sy & sx \cdot cy & cx \cdot cy \end{bmatrix}$$

where

$$\begin{aligned} cx &= \cos \theta_x & sx &= \sin \theta_x \\ cy &= \cos \theta_y & sy &= \sin \theta_y \\ cz &= \cos \theta_z & sz &= \sin \theta_z \end{aligned}$$

The rotation matrix $\mathbf{R}_{W \rightarrow T}$ (also known as the Direction Cosine Matrix (DCM) is given by:

$$\mathbf{R}_{W \rightarrow T} = \mathbf{R}_{T \rightarrow W}' = (\mathbf{R}_z \mathbf{R}_y \mathbf{R}_x)' = \mathbf{R}_x' \mathbf{R}_y' \mathbf{R}_z'$$

$$\mathbf{R}_{T \rightarrow W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx & sx \\ 0 & -sx & cx \end{bmatrix} \bullet \begin{bmatrix} cy & 0 & -sy \\ 0 & 1 & 0 \\ sy & 0 & cy \end{bmatrix} \bullet \begin{bmatrix} cz & sz & 0 \\ -sz & cz & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(rotation $\mathbf{R}_x'$) (rotation $\mathbf{R}_y'$) (rotation $\mathbf{R}_z'$)

$$\mathbf{R}_{W \rightarrow T} = \begin{bmatrix} cy \cdot cz & cy \cdot sz & -sy \\ sx \cdot sy \cdot cz - cx \cdot sz & sx \cdot sy \cdot sz + cx \cdot cz & sx \cdot cy \\ cx \cdot sy \cdot cz + sx \cdot sz & cx \cdot sy \cdot sz - sx \cdot cz & cx \cdot cy \end{bmatrix}$$

The derivatives of $\mathbf{R}_{W \rightarrow T}$ (used in the analytic Jacobian) are:

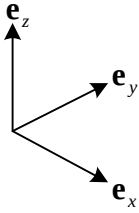
$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} = \begin{bmatrix} 0 & 0 & 0 \\ CX \cdot SY \cdot CZ + SX \cdot SZ & CX \cdot SY \cdot SZ - SX \cdot CZ & CX \cdot CY \\ -SX \cdot SY \cdot CZ + CX \cdot SZ & -SX \cdot SY \cdot SZ - CX \cdot CZ & -SX \cdot CY \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} = \begin{bmatrix} -SY \cdot CZ & -SY \cdot SZ & -CY \\ SX \cdot CY \cdot CZ & SX \cdot CY \cdot SZ & -SX \cdot SY \\ CX \cdot CY \cdot CZ & CX \cdot CY \cdot SZ & -CX \cdot SY \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} = \begin{bmatrix} -CY \cdot SZ & CY \cdot CZ & 0 \\ -SX \cdot SY \cdot SZ - CX \cdot CZ & SX \cdot SY \cdot CZ - CX \cdot SZ & 0 \\ -CX \cdot SY \cdot SZ + SX \cdot CZ & CX \cdot SY \cdot CZ + SX \cdot SZ & 0 \end{bmatrix}$$

Euler Rate $\leftrightarrow$ Body Rate Transforms

The effect of the Euler rotation sequence on the inertial cartesian frame vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ is as follows (initially, $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are mutually orthogonal):



- 1) $\mathbf{e}_x$ and $\mathbf{e}_y$ are rotated about $\mathbf{e}_z$ by θ_z to become $\mathbf{R}_z' \mathbf{e}_x$ and $\mathbf{R}_z' \mathbf{e}_y$. Euler rate $\dot{\theta}_z$ is defined relative to $\mathbf{e}_z$.
- 2) $\mathbf{R}_z' \mathbf{e}_x$ and $\mathbf{e}_z$ are rotated about $\mathbf{R}_z' \mathbf{e}_y$ by θ_y to become $\mathbf{R}_y' \mathbf{R}_z' \mathbf{e}_x$ and $\mathbf{R}_y' \mathbf{e}_z$. Euler rate $\dot{\theta}_y$ is defined relative to $\mathbf{R}_z' \mathbf{e}_y$.
- 3) $\mathbf{R}_z' \mathbf{e}_y$ and $\mathbf{R}_y' \mathbf{e}_z$ are rotated about $\mathbf{R}_y' \mathbf{R}_z' \mathbf{e}_x$ by θ_x to become $\mathbf{R}_x' \mathbf{R}_z' \mathbf{e}_y$ and $\mathbf{R}_x' \mathbf{R}_y' \mathbf{e}_z$. Euler rate $\dot{\theta}_x$ is defined relative to $\mathbf{R}_y' \mathbf{R}_z' \mathbf{e}_x$.

Summarizing the above in a table,

Euler rate	Axis of definition	Final axis location	Rotations incurred
$\dot{\theta}_z$	$\mathbf{e}_z$	$\mathbf{R}_x' \mathbf{R}_y' \mathbf{e}_z$	$\mathbf{R}_x' \mathbf{R}_y'$
$\dot{\theta}_y$	$\mathbf{R}_z' \mathbf{e}_y$	$\mathbf{R}_x' \mathbf{R}_z' \mathbf{e}_y$	$\mathbf{R}_x'$
$\dot{\theta}_x$	$\mathbf{R}_y' \mathbf{R}_z' \mathbf{e}_x$	$\mathbf{R}_y' \mathbf{R}_z' \mathbf{e}_x$	$\mathbf{I}$

So, each Euler rate must be projected onto the body axis with different transforms depending on the rotations incurred, as follows (note the use of the notation $\dot{\theta}^B$ and $\dot{\theta}^E$ to denote body rate and Euler rates, respectively):

$$\begin{bmatrix} \dot{\theta}_x^B \\ \dot{\theta}_y^B \\ \dot{\theta}_z^B \end{bmatrix} = \mathbf{I} \cdot \begin{bmatrix} \dot{\theta}_x^E \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_x' \begin{bmatrix} 0 \\ \dot{\theta}_y^E \\ 0 \end{bmatrix} + \mathbf{R}_x' \mathbf{R}_y' \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_z^E \end{bmatrix} = \left[\mathbf{I} \mathbf{S}_x + \mathbf{R}_x' \mathbf{S}_y + \mathbf{R}_x' \mathbf{R}_y' \mathbf{S}_z \right] \cdot \begin{bmatrix} \dot{\theta}_x^E \\ \dot{\theta}_y^E \\ \dot{\theta}_z^E \end{bmatrix}$$

where column selector matrices $\mathbf{S}_x$, $\mathbf{S}_y$, and $\mathbf{S}_z$ are defined as

$$\mathbf{S}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S}_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S}_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{R}_{EB} &\equiv \left[\mathbf{I} \mathbf{S}_x + \mathbf{R}_x' \mathbf{S}_y + \mathbf{R}_x' \mathbf{R}_y' \mathbf{S}_z \right] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx & sx \\ 0 & -sx & cx \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx & sx \\ 0 & -sx & cx \end{bmatrix} \cdot \begin{bmatrix} cy & 0 & -sy \\ 0 & 1 & 0 \\ sy & 0 & cy \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad \mathbf{I} \quad \mathbf{S}_x \quad \mathbf{R}_x' \quad \mathbf{S}_y \quad \mathbf{R}_x' \quad \mathbf{R}_y' \quad \mathbf{S}_z \end{aligned}$$

$\mathbf{R}_{EB} = \begin{bmatrix} 1 & 0 & -sy \\ 0 & cx & sx \cdot cy \\ 0 & -sx & cx \cdot cy \end{bmatrix}$	Euler rate to body rate transform
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The transform from body rate to Euler rate $\mathbf{R}_{BE}$ can be found by solving $\mathbf{R}_{BE} \cdot \mathbf{R}_{EB} = \mathbf{I}$ for $\mathbf{R}_{BE}$ (details omitted):

$\mathbf{R}_{BE} = \begin{bmatrix} 1 & sx \cdot ty & cx \cdot ty \\ 0 & cx & -sx \\ 0 & sx/cy & cx/cy \end{bmatrix}$	body rate to Euler rate transform
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Euler Accel $\leftrightarrow$ Body Accel Transforms

To derive the transform for Euler accel to body accel, differentiate the Euler rate to body rate expression $\dot{\theta}^B = \mathbf{R}_{EB} \dot{\theta}^E$:

$$\ddot{\theta}^B = \frac{\partial}{\partial t} (\mathbf{R}_{EB} \dot{\theta}^E) = \mathbf{R}_{EB} \ddot{\theta}^E + \dot{\mathbf{R}}_{EB} \dot{\theta}^E$$

$$\begin{aligned}\dot{\mathbf{R}}_{EB} &= \frac{\partial}{\partial t} \begin{bmatrix} 1 & 0 & -sy \\ 0 & cx & sx \cdot cy \\ 0 & -sx & cx \cdot cy \end{bmatrix} = \begin{bmatrix} 0 & 0 & -cy \cdot \dot{\theta}_y^E \\ 0 & -sx \cdot \dot{\theta}_x^E & cx \cdot cy \cdot \dot{\theta}_x^E - sx \cdot sy \cdot \dot{\theta}_y^E \\ 0 & -cx \cdot \dot{\theta}_x^E & -sx \cdot cy \cdot \dot{\theta}_x^E - cx \cdot sy \cdot \dot{\theta}_y^E \end{bmatrix} \\ \dot{\mathbf{R}}_{EB} \dot{\theta}^E &= \begin{bmatrix} 0 & 0 & -cy \cdot \dot{\theta}_y^E \\ 0 & -sx \cdot \dot{\theta}_x^E & cx \cdot cy \cdot \dot{\theta}_x^E - sx \cdot sy \cdot \dot{\theta}_y^E \\ 0 & -cx \cdot \dot{\theta}_x^E & -sx \cdot cy \cdot \dot{\theta}_x^E - cx \cdot sy \cdot \dot{\theta}_y^E \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_x^E \\ \dot{\theta}_y^E \\ \dot{\theta}_z^E \end{bmatrix} \\ \Rightarrow \dot{\mathbf{R}}_{EB} \dot{\theta}^E &= \begin{bmatrix} -cy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -sx \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + cx \cdot cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E - sx \cdot sy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -cx \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E - sx \cdot cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E - cx \cdot sy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \end{bmatrix}\end{aligned}$$

Euler accel to body accel transform $\ddot{\theta}^B = \mathbf{R}_{EB} \ddot{\theta}^E + \dot{\mathbf{R}}_{EB} \dot{\theta}^E$:

$$\begin{bmatrix} \ddot{\theta}_x^B \\ \ddot{\theta}_y^B \\ \ddot{\theta}_z^B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -sy \\ 0 & cx & sx \cdot cy \\ 0 & -sx & cx \cdot cy \end{bmatrix} \cdot \begin{bmatrix} \ddot{\theta}_x^E \\ \ddot{\theta}_y^E \\ \ddot{\theta}_z^E \end{bmatrix} + \begin{bmatrix} -cy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -sx \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + cx \cdot cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E - sx \cdot sy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -cx \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E - sx \cdot cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E - cx \cdot sy \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \end{bmatrix}$$

To derive the transform for body accel to Euler accel, differentiate the body rate to Euler rate expression $\dot{\theta}^E = \mathbf{R}_{BE} \dot{\theta}^B$:

$$\begin{aligned}\dot{\theta}^E &= \frac{\partial}{\partial t} (\mathbf{R}_{BE} \dot{\theta}^B) = \mathbf{R}_{BE} \ddot{\theta}^B + \dot{\mathbf{R}}_{BE} \dot{\theta}^B = \mathbf{R}_{BE} \ddot{\theta}^B + \dot{\mathbf{R}}_{BE} \mathbf{R}_{EB} \dot{\theta}^E \\ \dot{\mathbf{R}}_{BE} &= \frac{\partial}{\partial t} \begin{bmatrix} 1 & sx \cdot ty & cx \cdot ty \\ 0 & cx & -sx \\ 0 & sx/cy & cx/cy \end{bmatrix} = \begin{bmatrix} 0 & cx \cdot ty \cdot \dot{\theta}_x^E + \frac{sx}{cy^2} \cdot \dot{\theta}_y^E & -sx \cdot ty \cdot \dot{\theta}_x^E + \frac{cx}{cy^2} \cdot \dot{\theta}_y^E \\ 0 & -sx \cdot \dot{\theta}_x^E & -cx \cdot \dot{\theta}_x^E \\ 0 & \frac{cx \cdot cy}{cy^2} \cdot \dot{\theta}_x^E + \frac{sx \cdot sy}{cy^2} \cdot \dot{\theta}_y^E & \frac{-sx \cdot cy}{cy^2} \cdot \dot{\theta}_x^E + \frac{cx \cdot sy}{cy^2} \cdot \dot{\theta}_y^E \end{bmatrix} \\ \Rightarrow \dot{\mathbf{R}}_{BE} &= \begin{bmatrix} 0 & cx \cdot ty & -sx \cdot ty \\ 0 & -sx & -cx \\ 0 & cx/cy & -sx/cy \end{bmatrix} \cdot \dot{\theta}_x^E + \begin{bmatrix} 0 & \frac{sx}{cy^2} & \frac{cx}{cy^2} \\ 0 & 0 & 0 \\ 0 & \frac{sx \cdot sy}{cy^2} & \frac{cx \cdot sy}{cy^2} \end{bmatrix} \cdot \dot{\theta}_y^E \\ \dot{\mathbf{R}}_{BE} \cdot \mathbf{R}_{EB} &= \begin{bmatrix} 0 & cx \cdot ty & -sx \cdot ty \\ 0 & -sx & -cx \\ 0 & cx/cy & -sx/cy \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -sy \\ 0 & cx & sx \cdot cy \\ 0 & -sx & cx \cdot cy \end{bmatrix} \cdot \dot{\theta}_x^E \\ &+ \begin{bmatrix} 0 & \frac{sx}{cy^2} & \frac{cx}{cy^2} \\ 0 & 0 & 0 \\ 0 & \frac{sx \cdot sy}{cy^2} & \frac{cx \cdot sy}{cy^2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -sy \\ 0 & cx & sx \cdot cy \\ 0 & -sx & cx \cdot cy \end{bmatrix} \cdot \dot{\theta}_y^E\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 0 & cx^2 \cdot ty + sx^2 \cdot ty & sx \cdot cx \cdot cy \cdot ty - sx \cdot cx \cdot cy \cdot ty \\ 0 & -sx \cdot cx + sx \cdot cx & -sx^2 \cdot cy - cx^2 \cdot cy \\ 0 & \frac{cx^2}{cy} + \frac{sx^2}{cy} & sx \cdot cx - sx \cdot cx \end{bmatrix} \cdot \dot{\theta}_x^E \\
&+ \begin{bmatrix} 0 & \frac{sx \cdot cx}{cy^2} - \frac{sx \cdot cx}{cy^2} & \frac{sx^2}{cy} + \frac{cx^2}{cy} \\ 0 & 0 & 0 \\ 0 & \frac{sx \cdot cx \cdot sy}{cy^2} - \frac{sx \cdot cx \cdot sy}{cy^2} & \frac{sx^2 \cdot sy}{cy} + \frac{cx^2 \cdot sy}{cy} \end{bmatrix} \cdot \dot{\theta}_y^E \\
&= \begin{bmatrix} 0 & ty & 0 \\ 0 & 0 & -cy \\ 0 & \frac{1}{cy} & 0 \end{bmatrix} \cdot \dot{\theta}_x^E + \begin{bmatrix} 0 & 0 & \frac{1}{cy} \\ 0 & 0 & 0 \\ 0 & 0 & ty \end{bmatrix} \cdot \dot{\theta}_y^E = \begin{bmatrix} 0 & ty \cdot \dot{\theta}_x^E & \frac{1}{cy} \cdot \dot{\theta}_y^E \\ 0 & 0 & -cy \cdot \dot{\theta}_y^E \\ 0 & \frac{1}{cy} \cdot \dot{\theta}_x^E & ty \cdot \dot{\theta}_y^E \end{bmatrix} \\
&\mathbf{\dot{R}}_{BE} \mathbf{R}_{EB} \dot{\theta}^E = \begin{bmatrix} 0 & ty \cdot \dot{\theta}_x^E & \frac{1}{cy} \cdot \dot{\theta}_y^E \\ 0 & 0 & -cy \cdot \dot{\theta}_y^E \\ 0 & \frac{1}{cy} \cdot \dot{\theta}_x^E & ty \cdot \dot{\theta}_y^E \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_x^E \\ \dot{\theta}_y^E \\ \dot{\theta}_z^E \end{bmatrix} = \begin{bmatrix} ty \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + \frac{1}{cy} \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E \\ \frac{1}{cy} \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + ty \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \end{bmatrix}
\end{aligned}$$

Body accel to Euler accel transform $\ddot{\theta}^E = \mathbf{R}_{BE} \ddot{\theta}^B + \mathbf{\dot{R}}_{BE} \mathbf{R}_{EB} \dot{\theta}^E$:

$$\begin{bmatrix} \ddot{\theta}_x^E \\ \ddot{\theta}_y^E \\ \ddot{\theta}_z^E \end{bmatrix} = \begin{bmatrix} 1 & sx \cdot ty & cx \cdot ty \\ 0 & cx & -sx \\ 0 & sx/cy & cx/cy \end{bmatrix} \cdot \begin{bmatrix} \ddot{\theta}_x^B \\ \ddot{\theta}_y^B \\ \ddot{\theta}_z^B \end{bmatrix} + \begin{bmatrix} ty \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + \frac{1}{cy} \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \\ -cy \cdot \dot{\theta}_x^E \cdot \dot{\theta}_z^E \\ \frac{1}{cy} \cdot \dot{\theta}_x^E \cdot \dot{\theta}_y^E + ty \cdot \dot{\theta}_y^E \cdot \dot{\theta}_z^E \end{bmatrix}$$